Solvable Groups

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Introduction

The goal of this paper is to introduce the concept of a *solvable group*. Solvable groups have many applications, including applications in Galois theory, where solvable groups are used to prove that, in general, polynomial equations of degree greater than four are not solvable using a finite number of additions, multiplications, and root extractions. However, in order to investigate solvable groups we must first develop two related concepts of *commutator subgroups* and *metabelian groups*.

Before we begin, we will need some easy definitions.

Definition 1. Let $N \subseteq G$ be groups, and $g \in G$. A *left coset* gN and *right coset* Ng of N in G are defined, respectively, as

$$gN = \{gn \mid n \in N\} \quad \text{and} \quad Ng = \{ng \mid n \in N\}$$

Definition 2. Let G be a group. A *normal subgroup* N of G, written $N \triangleleft G$, is a subgroup of G such that for all $g \in G$, $gN = Ng$.

Proposition 1. Let G be a group. A subgroup N of G is normal if and only if for every $g \in G$ and $n \in N$, $gng^{-1} \in N$.

Proposition 2. Every subgroup of an abelian group G is normal in G.

Lemma. *Let G be a group and N a normal subgroup of G. Then $gN = N$ implies $g \in N$.*

Proof. Suppose $gN = N$. Then $\{gn \mid n \in N\} = \{n \mid n \in N\}$. Therefore, for the sets to be equal, there must be an $n \in N$ such that $g1 = n$, since $1 \in N$. Because $g = g1$ and $n \in N$, $g \in N$. $\square$

Definition 3. Let G be a group and N a normal subgroup of G. The *quotient group* of G with N, written G/N, is the set of all cosets of N in G under the operation $(uN)(vN) = (uv)N$ for all $u, v \in G$. Note that the identity of G/N is simply N.

With these definitions in mind, we are ready to proceed to our investigation of commutator subgroups.

Commutator Subgroups

Commutation is an interesting operation to study because it allows us to measure how "close" groups are to being abelian. If elements of a group commute, then the commutativity is reflected in the structure of that group's commutator subgroup, and vice versa. As a group's closeness to being abelian will enable us to study its solvability, we are interested in determining the structure of commutator subgroups. The goal of this section is to prove a theorem relating commutator subgroups to quotient groups that we will need later.

Definition 4. Let G be a group. The *commutation* of two elements $a, b \in G$ is the element $aba^{-1}b^{-1}$. The commutation of two elements is a *commutator*.

Definition 5. Let G be a group. The *commutator subgroup* G' of G is defined as

$$G' = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle.$$

Note that the commutator subgroup is the group generated by all the commutators of G.

Theorem 1. *Let G be a group and N a normal subgroup of G. Then G/N is abelian if and only if $G' \subseteq N$.*

Proof. We first prove the only if direction. Suppose G/N is abelian. We wish to show that $G' \subseteq N$. Let $x \in G'$ be a generator of G', so that $x = aba^{-1}b^{-1}$ for some $a, b \in G$. Because G/N is a quotient group, we know that $aN, bN \in G/N$. Therefore, we have the following string of equalities:

$$\begin{aligned} xN = (aba^{-1}b^{-1})N &= (aN)(bN)(a^{-1}N)(b^{-1}N) && \text{since } N \text{ is normal} \\ &= (bN)(aN)(a^{-1}N)(b^{-1}N) && \text{since } G/N \text{ is abelian} \\ &= (baa^{-1}b^{-1})N = N && \text{since } N \text{ is normal} \end{aligned}$$

Because $xN = N$, we have that $x \in N$, and thus $G' \subseteq N$.

We now prove the if direction. Suppose $G' \subseteq N$. We wish to show that G/N is abelian. Let $aN, bN \in G/N$. Since $a, b \in G$, we know that $aba^{-1}b^{-1} \in G'$ and therefore that $aba^{-1}b^{-1} \in N$. Likewise, we know that $bab^{-1}a^{-1} \in N$. Consider

$$(aN)(bN)(a^{-1}N)(b^{-1}N) = (aba^{-1}b^{-1})N = N$$

We are able to "factor out" the N from above because N is normal. We know that $(aba^{-1}b^{-1})N = N$ by the lemma because $aba^{-1}b^{-1} \in N$. Similarly, we can obtain

$$(bN)(aN)(b^{-1}N)(a^{-1}N) = (bab^{-1}a^{-1})N = N$$

As these two quantities are equal, we can equate them to get

$$(aN)(bN)(a^{-1}N)(b^{-1}N) = (bN)(aN)(b^{-1}N)(a^{-1}N)$$

Multiplying both sides by $(bN)(aN)$ gives

$$(aN)(bN)(a^{-1}N)(b^{-1}N)(bN)(aN) = (bN)(aN)(b^{-1}N)(a^{-1}N)(bN)(aN)$$

Because N is normal, we can rearrange terms and simplify to get

$$(ab)N = ((ba)N)(b^{-1}a^{-1}baN)$$

Because $b^{-1}a^{-1}ba \in N$ (since it is the commutation of b^{-1} and a^{-1} and hence is in $G' \subseteq N$) we know that $(b^{-1}a^{-1}ba)N = N$. Thus, we have

$$(ab)N = (ba)N$$

Therefore

$$(aN)(bN) = (ab)N = (ba)N = (bN)(aN)$$

And so G/N is abelian. $\square$

Metabelian Groups

Metabelian groups can be thought of as groups that are "close" to being abelian, in the sense that every abelian group is metabelian, but not every metabelian group is abelian. This closeness is reflected in the particular structure of their commutator subgroups. As we have developed techniques for examining commutator subgroups, we are now able to apply the techniques to examine this particular class of groups. Because, as we will see later, metabelian groups are very simple instances of solvable groups, it is worthwhile for us to examine metabelian groups before we move on to studying solvable groups in general. The goal of this section is to prove several interesting properties of metabelian groups.

Definition 6. A group G is metabelian if there exists a normal subgroup $A \triangleleft G$ such that both A and G/A are abelian.

Proposition 3. Every abelian group is metabelian.

Proof. Let G be an abelian group. Then all subgroups of G are normal, and $G' = 1$. So by theorem 1, all of G's quotient groups are abelian, and thus G is metabelian. $\square$

We may now prove the following three theorems.

Theorem 2. *G is metabelian if and only if $G'' = 1$ (G'' is the commutator subgroup of G').*

Proof. We first prove the only if direction. Let G be a metabelian group; we will show that $G'' = 1$. Because G is metabelian, it has a normal abelian subgroup N, and G/N is abelian. Thus, by theorem 1, $G' \subseteq N$. Since N is abelian, $G' = 1$, and so $G'' = 1$.

We now prove the if direction. Let $G'' = 1$. We will show that G is metabelian. We will do this by first establishing the existence of a normal abelian subgroup of G and then by showing that G's quotient with that group is abelian. Note that if G is abelian, then G is metabelian by proposition 3. So, we have only to consider the case where G is not abelian, and hence $G' \neq 1$. Thus, assume that G is not abelian.

We will now show the existence of a normal abelian subgroup of G. Consider the commutation of two elements $x, y \in G'$. We know that $xyx^{-1}y^{-1} = 1$ since $G'' = 1$ and so we have that $xy = yx$. Thus, arbitrary elements $x, y \in G'$ commute, so $G' \neq 1$ is an abelian subgroup of G. We must now show that G' is normal in G to establish the existence of a normal abelian subgroup of G. Let $g \in G$ and $h \in G'$. To show normality by proposition 1 we must show that $ghg^{-1} \in G'$. We know that $ghg^{-1}h^{-1} \in G'$ because $ghg^{-1}h^{-1}$ is the commutation of $g, h \in G'$. Because G' is a group containing h and $ghg^{-1}h^{-1}$, multiplying $ghg^{-1}h^{-1}$ by h yields another element in G'. Thus, $ghg^{-1}h^{-1}h = ghg^{-1} \in G'$, and so G' is normal in G. Thus, G' is a normal abelian subgroup of G.

We have established the existence of a normal abelian subgroup of G, namely G', so all that remains is to show that G/G' is abelian. Because $G' \triangleleft G$ and $G' \subseteq G'$, by theorem 1 we have that G/G' is abelian. Thus, G has an abelian normal subgroup, G', and G/G' is abelian, so G is metabelian. □

Theorem 3. *If H is a subgroup of a metabelian group G, then H is metabelian.*

Proof. Let H be a subgroup of the metabelian group G. Since G is metabelian, by theorem 2, $G'' = 1$. Consider H'. As H is a subgroup of G, it must be the case that H' is a subgroup of G'. To see this, let x be a generator of H', so that $x = aba^{-1}b^{-1} \in H'$ for some $a, b \in H$. Since $H \subseteq G$, $a, b \in G$ and hence $aba^{-1}b^{-1} = x \in G'$. Therefore, H' is a subgroup of G'. Thus, it must also be the case that H'' is a subgroup of G''. Since $G'' = 1$, it follows that $H'' = 1$. Thus, by theorem 2, H is metabelian. □

Theorem 4. *If G is metabelian and $\varphi : G \to K$ is a group homomorphism, then $\varphi(G)$ is metabelian.*

Proof. Let G be metabelian and φ be as above. We will show that $\varphi(G)'' = 1$, so that $\varphi(G)$ is metabelian by theorem 2.

To that end, we will show that commutation is respected by group homomorphisms in general. In other words, we will show that the image of a commutator subgroup is the commutator subgroup of the image. Let A and B be groups, and let X be a subgroup of A; let $\psi : A \to B$ be a group homomorphism. We will show that $\psi(X)' = \psi(X')$ by showing that $\psi(X)' \subseteq \psi(X')$ and $\psi(X') \subseteq \psi(X)'$. Note that the notation $\psi(X)'$ denotes the commutator subgroup of $\psi(X)$. Let $\alpha \in \psi(X)'$ be a generator of $\psi(X)'$. Then $\alpha = \psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1}$ for some $a, b \in X$. Because ψ is a group homomorphism, we may simply the expression to get $\alpha = \psi(aba^{-1}b^{-1}) \in \psi(X')$, as $aba^{-1}b^{-1} \in X'$. Thus, $\psi(X)' \subseteq \psi(X')$. Now, let $\beta \in \psi(X')$ be a generator of $\psi(X')$. Then $\beta = \psi(aba^{-1}b^{-1})$ for some $a, b \in X$. As ψ is a group homomorphism, we may expand this expression to get $\beta = \psi(a)\psi(b)\psi(a)^{-1}\psi(b)^{-1} \in \psi(X)'$, as β is the commutation of two elements $\psi(a), \psi(b) \in \psi(X)$. Thus, $\psi(X') \subseteq \psi(X)'$.

Therefore, $\psi(X)' = \psi(X')$. For our particular case, this means that for a subgroup H of G, that $\varphi(H)' = \varphi(H')$.

Thus, have that $\varphi(G)'' = \varphi(G')' = \varphi(G'')$. Since G is metabelian, $G'' = 1$, and as group homomorphisms always map identity to identity, we have that $\varphi(G)'' = \varphi(G'') = \varphi(1) = 1$. Thus, by theorem 2, $\varphi(G)$ is metabelian. $\square$

Solvable Groups

At last we are ready to define solvable groups.

Definition 7. A group G is solvable if there exist normal subgroups

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G,$$

such that for $i = 1, 2, \ldots, n$, the quotient group G_{i+1}/G_i is abelian.

We can now see that metabelian groups really are simple instances of solvable groups. A metabelian group is a solvable group which has the sequence of subgroups $1 = G_0 \subseteq G' \subseteq G_2 = G$.