Addendum

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Abstract

This document describes further results related to my dissertation.

1 Higher-order logic as a query language

Recall that the main open conjecture of chapter 3 is the semantics preservation of the translation $[] : HOL \to NRC$ for every domain-independent HOL sequent $\Gamma \vdash e : t$

$$[\![\Gamma \vdash e : t]\!] = [\![[\Gamma \vdash e : t]]\!]$$

This conjecture is still open, but there are two new results to report:

1. This conjecture is false for the internal language of a topos (defined in section 3.2.2).
2. This conjecture is false for HOL + weakening.

1.1 Topoi

The following theorems are proved in Coq. Let L denote the internal language of a topos, as defined in section 3.2.2. Our translation $[] : HOL \to NRA$ extends directly to $[] : L \to NRA$.

Theorem (Topos-Eta)**.**

$$[\![[id]]\!] = [\![[\Lambda ev]]\!]$$

Theorem (Topos-Beta-Weak1)**.** *Let $A \times B : f : \Omega$ be a term in the internal language of a topos. Then*

$$[\![[\langle id, y\rangle; f]]\!] = [\![[\langle \Lambda f, y\rangle; ev]]\!]$$

Theorem (Topos-Beta-Weak2)**.** *Let $A \times B : f : \Omega$ be a term in the internal language of a topos. Then*

$$\forall I \in [\![A]\!], J \in [\![B]\!], \quad (I, J) \in [\![f]\!] \quad implies \quad atoms(J) \subseteq atoms(I)$$

implies

$$[\![[\langle \pi_1; \Lambda f, \pi_2\rangle; ev]]\!] = [\![[f]]\!]$$

The condition above guarantees that $[\![\Lambda f]\!]$ will contain no constants not in $[\![f]\!]$. If we violate this condition, the theorem is not true:

Theorem (Topos-No-Beta)**.** *There exists a hereditarily domain-independent f, such as $1 \times D : \top : \Omega$, st*

$$[\![[\langle \pi_1; \Lambda f, \pi_2\rangle; ev]]\!] \neq [\![[f]]\!]$$

Using the above, and that $[\![\langle \pi_1; \Lambda f, \pi_2\rangle; ev]\!] = [\![f]\!]$, and that $[]$ is semantics preserving for hereditarily domain-independent terms, we obtain that

Theorem (Topos-No-Sem)**.** *There exists a domain independent f, such as $\langle \pi_1; \Lambda\top, \pi_2\rangle; ev$, such that*

$$[\![f]\!] \neq [\![[f]]\!]$$

1.2 HOL + weakening

We can re-cast the development of the previous section in terms of HOL, if we add an additional typing rule:

$$\text{WEAKEN}\quad \frac{\Gamma \vdash e : t \qquad x\ fresh}{\Gamma, x : s \vdash e : t} \qquad \text{WEAKEN-TRANS}\quad \frac{[\Gamma \vdash e : t] = [\Gamma] \vdash e' : [t]}{[\Gamma, x : s \vdash e : t] = [\Gamma], x : [s] \vdash e' : [t]} \qquad \text{WEAKEN-SEM}\quad [\![\Gamma, x : s \vdash e : t]\!] = \pi_1 ; [\![\Gamma \vdash e : t]\!]$$

Whereas in HOL, every sequent corresponds to a unique typing derivation, and vice-versa, with the addition of weakening, HOL sequents can have more than one derivation. Since the meaning of a sequent is a function of its typing derivation, HOL sequents can now potentially have more than one meaning; i.e., $[\![]\!]$ must be defined as a relation. Typically, we would prove a *coherence* theorem for HOL+weakening, stating that $[\![]\!]$ is a functional relation; i.e., regardless of which derivation we use to compute the meaning of a sequent, the meaning will be the same. We would need to use the coherence theorem to prove, for example, that the translation $[] : HOL + weakening \to NRC$ is semantics preserving for hereditarily domain-independent derivations. However, for the purposes of showing that $[] : HOL + weakening \to NRC$ is not semantics preserving for domain-independent terms, all we need to do is exhibit a counter-example derivation.

Consider:

$$\cfrac{\cfrac{\cfrac{y : D \vdash \top : 2}{\vdash \lambda y : D.\top : D \to 2}\ \text{ABS}}{x : D \vdash \lambda y : D.\top : D \to 2}\ \text{WEAKEN} \qquad \cfrac{}{x : D \vdash x : D}\ \text{VAR1}}{x : D \vdash (\lambda y : D.\top) x : D \to 2}\ \text{APP}$$

Setting $[\![D]\!] = \{c\}$, we find that (ignoring superfluous units) :

$$\cfrac{\cfrac{\cfrac{[\![[y : D \vdash \top : 2]]\!](c \mapsto \top)}{[\![[\vdash \lambda y : D.\top : D \to 2]]\!](() \mapsto \{\})}\ \text{ABS}}{[\![[x : D \vdash \lambda y : D.\top : D \to 2]]\!](c \mapsto \{\})}\ \text{WEAKEN} \qquad \cfrac{}{[\![[x : D \vdash x : D]]\!](c \mapsto c)}\ \text{VAR1}}{[\![[x : D \vdash (\lambda y : D.\top) x : D \to 2]]\!](c \mapsto \bot)}\ \text{APP}$$

We conclude that $[\![[y : D \vdash \top : 2]]\!](c \mapsto \top)$ and $[\![[x : D \vdash (\lambda y : D.\top) x : D \to 2]]\!](c \mapsto \bot)$. As before, if we assume $[]$ is semantics preserving for domain-independent terms, we have a contradiction by noting that $[\![y : D \vdash \top : 2]\!] = [\![x : D \vdash (\lambda y : D.\top) x : D \to 2]\!]$ (beta) and $[\![[\top]]\!] = [\![\top]\!]$.

It is also the case that $[] : HOL + weakening \to NRC$ is incoherent: using a typing derivation that does not including weakening, we have that $[\![[x : D \vdash (\lambda y : D.\top) x : D \to 2]]\!](c \mapsto \top)$. This is why this counter-example does not apply to pure HOL.

Finally, it is worth noting that, if you consider $[\![]\!]$ as a translation from HOL to the internal language of a topos (L, as defined in 3.2.2), then there is no HOL sequent that maps to $\langle \pi_1 ; \Lambda f, \pi_2 \rangle ; ev$. To see this, note that the only way to obtain $\pi_1 ; \Lambda f$ is by rule VAR1, which implies that Λf must be a sequent of projections, which is impossible. Weakening allows us to obtain $\pi_1 ; \Lambda f$ by a rule other than VAR1 (namely, WEAKEN).

We conjecture that $[] : HOL + weakening \to NRC$ is both coherent and semantics preserving for hereditarily domain-independent terms.